



TITLE:

On Unorientable Surfaces in S^3 (結び目と3次元多様体)

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On Unorientable Surfaces in S^3

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§1. $Z/4$ -quadratic spaces.

We recall the definition of $Z/4$ -quadratic spaces. Let V be a finite dimensional vector space over $Z/2$ provided with a non-singular symmetric bilinear form $(x, y) \mapsto x \cdot y \in Z/2$, and let φ be a function : $V \longrightarrow Z/4$ satisfying $\varphi(x + y) = \varphi(x) + \varphi(y) + 2(x \cdot y)$ for all $x, y \in V$. φ is called a $Z/4$ -quadratic function and $X = (V, \cdot, \varphi)$ is called a $Z/4$ -quadratic space.

Definition. A $Z/4$ -quadratic space (V, \cdot, φ) is even, if $\varphi(x) \equiv 0 \pmod{2}$ for all $x \in V$.

A $Z/4$ -quadratic space (V, \cdot, φ) is odd, if $\varphi(x) \equiv 1 \pmod{2}$ for some $x \in V$.

(Even $Z/4$ -quadratic spaces are usually called $Z/2$ -quadratic spaces.)

Example. Let F be a smoothly imbedded (not necessarily orientable) surface in S^3 whose boundary ∂F is homeomorphic to S^1 . Then we can define a $Z/4$ -quadratic function $\varphi: H_1(F; Z/2) \longrightarrow Z/4$ as follows:

Let C be an immersed circle in F . The normal bundle ν_C of C in S^3 has a unique trivialization $\nu_C = S^1 \times R^2$ such that the linking number of $C = S^1 \times 0$ and $S^1 \times *$ ($* \in R^2, * \neq 0$) is zero. Since the normal bundle of C in F defines a sub-bundle ν of ν_C , we can count the number $n(C)$ of right-handed

half twists of ν , using the trivialization above. Now the required function φ is defined by

$$\varphi(C) = n(C) + 2 \text{ Self}(C) \pmod{4},$$

where $\text{Self}(C)$ is the number of the self-intersection points of C on F .

Proposition 1. ([5], Lemma 5.1) $\varphi(C) \in \mathbb{Z}/4$ depends only on the $\mathbb{Z}/2$ -homology class of C . The function $\varphi: H_1(F; \mathbb{Z}/2) \rightarrow \mathbb{Z}/4$ is $\mathbb{Z}/4$ -quadratic with respect to the $\mathbb{Z}/2$ -intersection pairing of $H_1(F; \mathbb{Z}/2)$.

Remark. Let X_F denote the $\mathbb{Z}/4$ -quadratic space $(H_1(F; \mathbb{Z}/2), \cdot, \varphi)$ above. Then X_F is even, if F is orientable, and X_F is odd, if F is unorientable.

In [2], E. H. Brown defined a generalized $\mathbb{Z}/8$ Arf invariant, called Brown's invariant, of $\mathbb{Z}/4$ -quadratic spaces. The Witt group W is isomorphic to $\mathbb{Z}/8$ by Brown's invariant. (See [5] for the definition of the Witt group.) The definition of Brown's invariant is as follows:

Let X be a $\mathbb{Z}/4$ -quadratic space (V, \cdot, φ) . We set

$$\lambda(X) = \sum_{x \in V} \sqrt{-1}^{\varphi(x)} \in \mathbb{C}.$$

Then the complex number $\lambda(X)$ has the property that $\lambda(X)^8 \in \mathbb{R}^+$, and the integer m modulo 8 is well-defined. It is called Brown's invariant and is denoted by $\beta(X) \in \mathbb{Z}/8$.

Proposition 2.A (2.B). The isomorphism classes of even (odd) $\mathbb{Z}/4$ -quadratic spaces can be completely classified by the dimension of V over $\mathbb{Z}/2$ and Brown's invariant $\beta(X)$.

For the proof, see [1], [2], and [5].

§2. Unorientable surfaces in S^3 .

Let us consider smoothly imbedded surfaces in S^3 . Two surfaces F and G are regular homotopic, if there is a continuous family $\{F_t\}$ $0 \leq t \leq 1$ of smoothly immersed surfaces in S^3 such that $F_0 = F$, $F_1 = G$. In [6], the author has classified orientable surfaces with boundary in S^3 by regular homotopy. (See also [4].) In this section we classify unorientable surfaces in S^3 whose boundaries are homeomorphic to S^1 by regular homotopy. See also [2] Example (1.28).

Theorem. Two smoothly imbedded (not necessarily orientable) surfaces F , G in S^3 whose boundaries are homeomorphic to S^1 are regular homotopic if and only if the associated $\mathbb{Z}/4$ -quadratic spaces X_F and X_G are isomorphic.

Corollary A (B). Two smoothly imbedded orientable (unorientable) surfaces F , G in S^3 whose boundaries are homeomorphic to S^1 are regular homotopic if and only if $\dim_{\mathbb{Z}/2} H_1(F; \mathbb{Z}/2) = \dim_{\mathbb{Z}/2} H_1(G; \mathbb{Z}/2)$ and $\beta(X_F) = \beta(X_G)$.

We prove Theorem for unorientable surfaces. See [6] for the proof of orientable surfaces. Let F and G be smoothly imbedded unorientable surfaces in S^3 whose boundaries are homeomorphic to S^1 such that X_F and X_G are isomorphic.

Lemma 1. Suppose that $\{e_1, \dots, e_r\}$ is a basis of $H_1(F; \mathbb{Z}/2)$ satisfying the condition (*);

$$(*) \quad e_i \cdot e_j = 0 \quad (i \neq j).$$

Then e_1, \dots, e_r can be represented by mutually disjoint imbedded circles c_1, \dots, c_r .

Remark. By the non-singularity of the intersection pairing of $H_1(F; \mathbb{Z}/2)$, the condition (*) implies $e_i \cdot e_i = 1 \in \mathbb{Z}/2$ for all $i = 1, \dots, r$, and therefore $\phi(e_i) = \pm 1 \in \mathbb{Z}/4$.

(proof of Lemma 1) Each $\mathbb{Z}/2$ -homology class e_i can be represented by a generic immersion of S^1 . Using the method illustrated in Figure 1, we may assume that the class e_i is represented by an imbedded circle c_i .

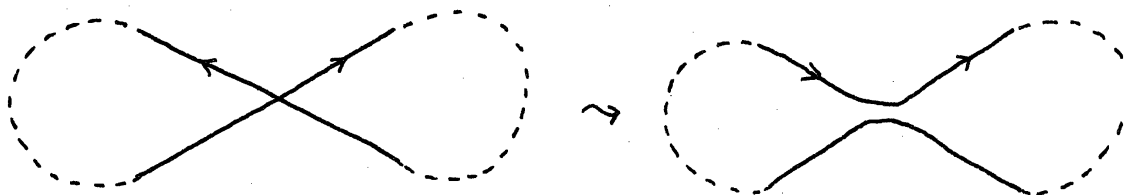
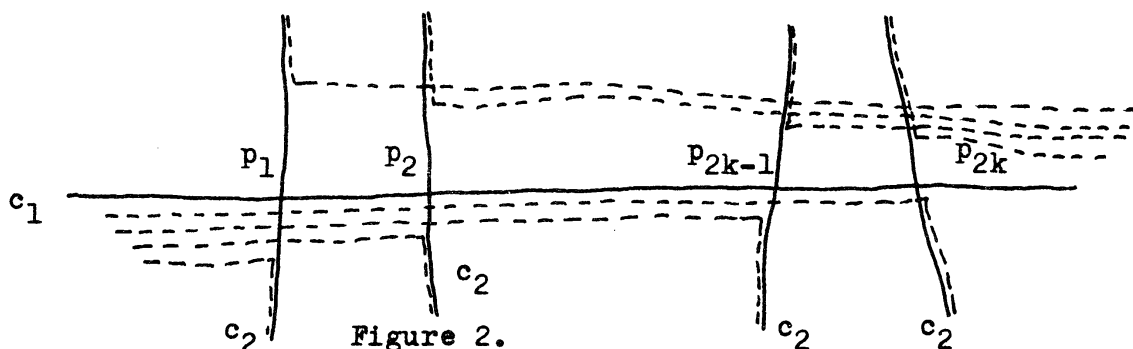


Figure 1.

Since we can prove this lemma by an induction on r , we shall prove in the case $r = 2$. Let c_1, c_2 be imbedded circles representing the elements e_1, e_2 . As $e_1 \cdot e_2 = 0 \in \mathbb{Z}/2$, $c_1 \cap c_2 = \{p_1, p_2, \dots, p_{2k-1}, p_{2k}\}$. If $k \neq 0$, we modify the curve c_2 as the dotted line in Figure 2. This can be done, because the regular neighborhood of the circle c_1 is a Möbius band. (See Remark above.) The new curve, also denoted by c_2 , has no intersection points with c_1 and represents the same $\mathbb{Z}/2$ -homology class e_2 as before, but it has some self-intersection points. Using the method illustrated in Figure 1 again, we kill these double points, and the lemma is proved.



From the classification of unorientable surfaces, the $\mathbb{Z}/2$ -vector space $H_1(F; \mathbb{Z}/2)$ has a basis $\{e_1, \dots, e_r\}$ which satisfies the condition (*). Let c_i be the imbedded circle in Lemma 1, and N_i be a regular neighborhood of c_i , for $i = 1, \dots, r$. Let N denote the boundary-connected-sum of N_i 's in F . Since the boundary ∂N_i of N_i is homeomorphic to S^1 , for $i = 1, \dots, r$, the boundary ∂N of N is also homeomorphic to S^1 , and $\partial(F - \text{int } N)$ is homeomorphic to $S^1 \cup S^1$ (disjoint union).

From the following Mayer-Vietoris exact sequence;

$$\begin{array}{ccccccc}
 0 \longrightarrow & H_1(\partial N; \mathbb{Z}) & \longrightarrow & H_1(N; \mathbb{Z}) \oplus H_1(F - \text{int} N; \mathbb{Z}) & \longrightarrow & H_1(F; \mathbb{Z}) & \xrightarrow{\quad} 0 \\
 & \parallel & & \parallel & & \parallel & \\
 & \mathbb{Z} & & r\mathbb{Z} & & r\mathbb{Z} & \\
 & & & & & & \\
 & \nearrow & & & & & \\
 0 \longrightarrow & H_0(\partial N; \mathbb{Z}) & \longrightarrow & H_0(N; \mathbb{Z}) \oplus H_0(F - \text{int} N; \mathbb{Z}) & \longrightarrow & H_0(F; \mathbb{Z}) & \longrightarrow 0 \\
 & \parallel & & \parallel & & \parallel & \\
 & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} &
 \end{array}$$

we obtain $H_i(F - \text{int} N; \mathbb{Z}) = \mathbb{Z}$ ($i=0,1$), and therefore $F - \text{int} N$ is homeomorphic to $S^1 \times [0, 1]$, and

Lemma 2. F is regular homotopic to N .

Since the $\mathbb{Z}/4$ -quadratic space X_G is isomorphic to X_F , there is a basis $\{f_1, \dots, f_r\}$ of $H_1(G; \mathbb{Z}/2)$ such that

$$\begin{aligned}
 e_i \cdot e_j &= f_i \cdot f_j & (i, j = 1, \dots, r) \\
 \varphi(e_i) &= \varphi(f_i) & (i = 1, \dots, r).
 \end{aligned}$$

Let d_i 's be mutually disjoint imbedded circles on G representing f_i 's as in Lemma 1, and let M denote the boundary-connected-sum of regular neighborhoods of d_i 's in G . Now from the equality $\varphi(e_i) = \varphi(f_i)$, it is easy to construct a regular homotopy between N and M ([3]), and therefore F and G are regular homotopic by Lemma 2. The converse is quite trivial and Theorem is proved.

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